**Introduction**

Standard buckling equations are commonly known. However, through FEM-development, analysis of post buckling behavior is possible and now elements in post buckling state can be used in construction design to the benefit of the constructions’ performance.

**Negative stiffness effect**

A buckled leaf spring comprises a negative stiffness in the lateral direction for the range:

\[-0.87u_x < x < 0.87u_x\]

With \(u_x = \sqrt{\frac{5}{3}u_x L}\)

Beyond this range the buckling is ‘transforming’ into a pure s-mode bending which is reached at: \(x = ±u_x\) With moving back the buckling does not re-occur. In combination with manufacturing tolerances-effects, the (JPE-) working range is:

\[-\frac{1}{2}u_x < x < \frac{1}{2}u_x\]

**Buckling force**

The buckling force can be considered constant after buckling and can be determined with:

\[F_{\text{buckling}} = 4\pi^2 \frac{E_l}{L^2}\]

**Linear stiffness**

The vertical (longitudinal) stiffness is zero. And as said, laterally it comprises a negative stiffness. The y-stiffness is similar to the transverse stiffness of a unbuckled leaf spring:

\[C_x = -44.4 \frac{E_l}{L^3}\]

\[C_y = \frac{E_l b^3}{L^3}\]

\[C_z = 0\]

**Stress**

The maximum (Von Mises) stress is located in the ‘bending poles’ of the buckling shape which are on \(\frac{1}{14}L\). For the location of the maximum stress, see the red spots in the picture. An approximation of the stress:

\[\sigma_{\text{max}} = 53 \cdot \frac{E_l}{L} \cdot \frac{u_z}{L} \cdot \sqrt{1 - \left(\frac{x}{0.87u_x}\right)^2}\]

(1st order estimate: please verify with FEM or consult JPE)

Thus \(\sigma_{\text{max}}\) decreases when moving in lateral direction: the buckling ‘gets relieved’. Maximum compression can be approximated with:

\[u_{z-\text{max}} = \frac{\sigma_{\text{allowable}} L^2}{53E_l}\]

(1st order estimate: please verify with FEM or consult JPE)

**Reaction Moment**

Obviously, to maintain the shape as depicted a moment at the top must be applied, this is qualified in the graph. This moment can be approximated with:

\[M_y = -3.2 E \cdot \frac{x^3}{L} \cdot \frac{u_z}{L} \cdot \sqrt{1 - \left(\frac{x}{0.87u_x}\right)^2}\]

(1st order estimate: please verify with FEM or consult JPE)

(negative implies pointing out of the sheet with right-hand-rule)

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**Rules of thumb**

- \(C_z\) is not affected by \(u_x\) or by \(u_z\)
- \(F_z\) is not affected by \(u_x\) or by \(u_z\)
- \(\sigma_{\text{max}}\) is affected with \(u_x\) and by \(u_x\)
- The range of motion can be increased with \(\sqrt{L}\) or \(\sqrt{u_z}\)
- The stress is at the ‘bending pole’ (see graphical layout);
  - movement towards the “belly” at \(\frac{1}{14}L\)
  - movement away at \(L - u_x - \frac{1}{14}L\)

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